ROM2F/2011/03

Gauge theories on Ω-backgrounds from non commutative Seiberg-Witten curves

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Abstract

We study the dynamics of a $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with fundamental or adjoint matter in presence of a non trivial Ω-background along a two dimensional plane. The prepotential and chiral correlators of the gauge theory can be obtained, via a saddle point analysis, from an equation which can be viewed as a non commutative version of the "standard" Seiberg and Witten curve.

Contents

1 Introduction and Summary

In [1, 2], Seiberg and Witten (SW) provided us with a beautiful solution for the quantum prepotential of a four-dimensional Yang-Mills gauge theory with $\mathcal{N} = 2$ supersymmetry (to be called $\mathcal{N} = 2$ gauge theory from now on) at the non perturbative level. In this set up, the dynamics of the gauge theory is encoded in the periods of a two-dimensional Riemann surface defined by a holomorphic curve.

With the introduction of localization techniques [3, 4, 5] in the study of non perturbative effects for such teories, the emergence of the curve and of the prepotential was directly derived from multi-instanton computations [6]. Localization requires a deformation of the spacetime geometry, the so called $\Omega_{\epsilon_1,\epsilon_2}$ background, regularizing the spacetime volume and leading to a finite multi instanton partition function $Z(\epsilon_1, \epsilon_2, q)$. The prepotential of the $\mathcal{N} = 2$ is identified with the free energy $\mathcal{F} = -\epsilon_1\epsilon_2 \ln Z$ in the limit $\epsilon_{1,2} \to 0$. The SW curves emerge from a saddle point evaluation of the multi instanton partition function in the limit $\epsilon_\ell \to 0$ [6]. Besides its regularization role, the parameters ϵ_ℓ can be interpreted in string theory as the vacuum expectation values of certain Ramond-Ramond fields [7]. Moreover the coefficients in the double ϵ expansion of $\mathcal{F}$ have been related [8] to the so called $\mathcal{F}_{g,n}$ topological string amplitudes first computed in [9].

It is natural to ask how the SW theory is modified in the presence of the $\Omega_{\epsilon_1,\epsilon_2}$ backgrounds. Here we address this question focusing on the simplest case $\epsilon_\ell = (0, \epsilon)$. This type of background has recently received a lot of attention due to its relations with quantum integrable systems [10].

The limit $\epsilon_1 \to 0$ is also interesting from the point of view of the so called AGT correspondence [11] since it corresponds to the quasiclassical limit where the central charge of the underlying CFT becomes very large.

The case of the $SU(N)$ theory with fundamental matter has been very recently studied in [12] where an " ϵ deformed SW curve" was derived from a saddle point analysis of the instanton partition function. The chiral correlators of the SYM theory were computed in terms of the integrals of $\lambda_J = x^J dz$ where $z(x)$ is a holomorphic function determined by the saddle point equations which, in the limit $\epsilon_\ell \to 0$, lead to the "standard" SW curve. In this paper we revisit these results and extend them to the case with adjoint matter where the ϵ deformed differential is found as a solution of an integral equation.

We rely on a saddle point analysis and encode the information about the saddle point solution into a single holomorphic function $z(x)$[1]. This function specifies the instanton distribution dominating the partition function in the limit $\epsilon_1 \to 0$. It is useful to think of gauge instantons as D(-1) branes bounded to D3 and D7 branes. After localization, the instantons distribute along the plane transverse to both the D3 and D7 branes at distances of order ϵ_ℓ from the positions of the D3 branes. In the limit $\epsilon_\ell \to 0$ they condense

[1] In the main text we work with the variable $w(x)$ connected to $z(x)$ via $w(x) = e^{-z(x)}$.

into continuous intervals centered at the D3 brane positions. One can think of this configuration as a two dimensional electrostatic system made out of metallic plates of charge -2 near D3 branes and point like charges +1 at the D7-brane positions[2]. The saddle point equations then become the conditions that the potential is constant along the metallic plates and the imaginary part of the holomorphic function $z(x)$ is identified with the electrostatic potential. Finding the SW curve is then equivalent to solve the electrostatic problem. Such techniques have also appeared in connection with matrix models in [13] and [14] for an earlier reference. The case of finite ϵ, can be thought of as a discretization of this electrostatic problem where the metallic plates split into infinite number of dipoles with dipole length ϵ. The saddle point equation once again is expressed as the condition that the potential at the center of any dipole coincide.

Alternatively the saddle point equations can be written as functional equations for $z(x)$. We will show that these equations in the case of fundamental matter can be thought of as a "non commutative" or "quantum" version of the SW curve. Evidences of such non commutative structure in the case of adjoint matter will be also presented. Our result provides a further support to the proposal in [10] for a relation of the gauge dynamics in the ϵ background to Toda and Calogero-Moser quantum integrable models.

To be concrete, let us consider a SW curve written as

$$W(x, e^z) = 0$$

and a one form differential $\lambda = x dz$. For a $SU(N)$ gauge theory with fundamental matter, W is a polynomial of order N in x and order two in e^z. We claim that the ϵ deformed dynamics is encoded into the non commutative version of the curve

$$W(\hat{x}, e^{\hat{z}})|\Psi\rangle = 0 \qquad [\hat{z}, \hat{x}] = \epsilon \tag{1}$$

This non commutative relation can be realized by taking either $z = \epsilon\partial_x$ or $x = -\epsilon\partial_z$. In the first case $|\Psi\rangle$ is realized as a function $\Psi(x)$. (1) becomes a difference equation relating $\Psi(x)$ to $e^{\pm\hat{z}}\Psi(x) = \Psi(x \pm \epsilon)$. Specifying to the case of fundamental matter we will show how this equation reproduces the one following from a saddle point analysis of the multi instanton partition function. In particular the deformed SW differential will be related to $\Psi(x)$

[2] See Appendix B and expecially (91) for a justification of this statement.

in a simple way. The difference equation will be solved and written in a continuous fraction form that will give the full ϵ dependence at each order in q. This generalizes a similar result in [12] were the $U(1)$ solution was written in terms of hypergeometric functions. In the case of adjoint matter, W is given in terms of theta functions and the analysis is more involved, since the order in e^z grows along with the q expansion. The resulting difference equation can be still solved order by order in q and we tested it against a direct multi instanton computation.

Alternatively (1) can be seen as a differential equation of order N for $\tilde{\Psi}(z) = |\Psi\rangle$ after the identification $\hat{x} = -\epsilon\partial_z$. This was the point of view taken in [15, 16, 17] for the pure gauge theory and [18] for the case of adjoint matter. The SW differential was identified with $\lambda = d\ln\tilde{\Psi}(z)$ and the periods where checked against the formulae for the leading ϵ corrections. In the case of an $SU(2)$ gauge theory, this leads [18, 19, 20, 21] to very robust tests of the correspondences between ϵ deformed gauge theories and CFT's or quantum integrable models. These results inspired our proposal. We remark that in this formulation the resulting N order differential equation can be typically solved only perturbatively in ϵ, in contrast with the difference equation which determines the full ϵ dependence of the differential. Clearly $\Psi(x)$ and $\tilde{\Psi}(z)$ are related to each other via a Fourier transform.

It would be nice to explore the implications of this non commutative structure in the M theory and type IIA descriptions of Seiberg-Witten theory. In IIA theory [22], the gauge theory is realized in terms of N D4 branes suspended between two NS5 branes. The endpoints of the D4 branes on the two NS5 branes behave as charges in an appropriate sense consistently with our electrostatic analogy. This picture lifts to M theory, where the brane system is replaced by a single M5 brane wrapping the two dimensional curve and the four dimensional spacetime. The eleventh dimensional circle is identified with the imaginary part of z which is compact due to the trivial identification $z \sim z + 2\pi i$. The holomorphicity of $z(x)$ ensures that the imaginary part of $z(x)$ admits the interpretation of a two dimensional electrostatic potential as claimed. Our results suggest that the ϵ deformation can be realized in these pictures by promoting the spacetime coordinates x, z to non commutative variables.

The paper is organized as follows: in section 2 we review the computation of the instanton partition function. In section 3 we use a saddle point analysis to evaluate such partition function and derive the general form of the saddle point equations, prepotential and chiral correlators of the gauge

theory. In section 4 and 5 we specify the results to the case of $SU(N)$ gauge theories with fundamental and adjoint matter respectively. In the Appendices we collect several useful technical data. In appendix A we discuss some properties of the counting function encoding the information about the saddle point solution. In Appendix B we present an alternative derivation of the saddle point equations based on the extremization over the so called profile function describing the shape of the Young tableaux which give the leading contribution to the instanton partition function. In Appendix C we present some tests of the ϵ deformed SW differentials against direct multi instanton computations. In Appendix D we comment on the connection to quantum integrable systems.

2 Instanton partition functions

In this section we review the computation of the instanton partition functions and of the chiral correlators for a $\mathcal{N}=2$ gauge theory with gauge group $SU(N)$ and matter in the fundamental or adjoint representation.

We start from $\mathcal{N}=4$ with gauge group $SU(N)$ realized in terms of open strings connecting N D3 branes in flat ten dimensional spacetime. After a mass deformation this becomes the so called $\mathcal{N}=2^*$ theory, i.e. $\mathcal{N}=2$ gauge theory with a massive adjoint hypermultiplet. In the limit of large mass it reduces to a pure $\mathcal{N}=2$ gauge theory. Alternatively, the adjoint matter can be projected out via an orbifold projection, i.e by modding out an internal four dimensional space by a discrete group. Fundamental matter can be added by including certain number of D7 branes.

The brane picture describes also the non perturbative objects of the gauge theory. Instantons of winding number k are realized by introducing k D(-1) branes. The instanton moduli are associated to the massless modes of those open strings which have at least one end on a D(-1) brane. The prepotential and chiral correlators of the gauge theory are computed by integrals over the resulting moduli space. The explicit evaluation of these integrals can be carried out with the help of localization techniques that reduce the integrations to the evaluation of a determinant at a finite number of isolated fixed points of the moduli space symmetries. To achieve complete localization, both gauge and Lorentz symmetries should be broken. Gauge symmetries can be broken by turning on a vacuum expectation value for the adjoint scalar field $\langle\Phi\rangle=\mathrm{diag}\{a_u\}$ in the SU(N) vector multiplet. Lorentz symme-

tries can be broken by turning on a non trivial background $\Omega_{\epsilon_1,\epsilon_2}$ on the four dimensional spacetime. The parameters $\{a_u, \epsilon_{1,2}\}$ parametrize the Cartan subgroup of the gauge and Lorentz symmetries, $SU(N) \times SO(4)$, broken by the background. The flat space result can be recovered by sending $\epsilon_{1,2} \to 0$ at the end of the computation.

The parameters a_u, $u = 1, \ldots, N$ we introduced before specify the positions of the D3 branes along a transverse plane. The positions of the instantons along this plane will be denoted by ϕ_I, $I = 1, ...k$. The $\Omega_{\epsilon_1,\epsilon_2}$ background creates a non trivial potential which penalizes those instantons trying to move away from the D3 brane and thus regularizing the space-time volume $\mathrm{vol}_{\mathbb{R}^4} \sim \frac{1}{\epsilon_1\epsilon_2}$. As a result, instantons distribute near the points a_u's where the D3 branes sit and the ϕ_I's can be accommodated in a Young tableau centered at a_u with boxes of sizes $\epsilon_{1,2}$. The instanton partition function can then be written as

$$\begin{aligned} Z_{\text{inst}}(q) &= \sum_{k=0}^{\infty} q^k Z_k = 1 + \sum_{k=1}^{\infty} q^k \int \frac{1}{k!} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i}\, z_k(\phi) \\ &= 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \int \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \prod_{I,J}^{k}{}' D(\phi_I - \phi_J) \prod_{I=1}^{k} Q_0(\phi_I) \\ &= \sum_{\vec{Y}} \prod_{I,J}^{k}{}' D(\phi_I^Y - \phi_J^Y) \prod_{I=1}^{k} q\, Q_0(\phi_I^Y) \end{aligned} \tag{2}$$

with $D(x)$, $Q_0(x)$ the contributions of open strings with both or a single end respectively on the D(-1) instantons. The prime in the product denotes the omission for $I = J$ given by the replacement $D(0) \to D'(0)$. The form of these functions depends on the specific matter content and will be given below in this section.

The integrals in (2) are evaluated by closing the contours in the complex plane and picking up the corresponding residues[3]. The relevant poles of the integrands (those whose contribution do not sum up to zero) (see (6),(8) below) are specified by the N Young tableaux set $\{Y_u\}$ with total number of boxes k. Explicitly

$$\phi_I^Y = \phi_{u,i_u,i'_u} = a_u + (i'_u - 1)\epsilon_1 + (i_u - 1)\epsilon_2 \tag{3}$$

[3]We take the pole prescription $\mathrm{Im}\epsilon_1 >> \mathrm{Im}\epsilon_2 > 0$ and close the contours in the upper half plane.

with i'_u, i_u running over the rows and columns of the Young tableau Y_u. The instanton partition function is then given by summing over all possible Young tableaux.

The partition function encodes the information about the gauge theory prepotential via the identification

$$\mathcal{F}(\epsilon_1, \epsilon_2, q) = \sum_{k=1}^{\infty} \mathcal{F}_k q^k = -\epsilon_1 \epsilon_2 \ln Z(q) \tag{4}$$

with $Z(q) = Z_{\rm pert}(\tau) Z_{\rm inst}(q)$ given in terms of the perturbative $Z_{\rm pert}(\tau)$ (tree level and one loop) and the instanton contribution $Z_{\rm inst}(q)$ given by (2). The details of $Z_{\rm pert}(\tau)$ will not be relevant to our analysis since $Z_{\rm pert}(\tau)$ does not depend on the details of the instanton configuration dominating the saddle point. The classical information of where the Young tableaux are located will be supplemented later by giving the periods, a_u, of the SW curve. An explicit form of $Z_{\rm pert}$ can be found in Appendix B. In the limit $\epsilon_\ell \to 0$, $\mathcal{F}$ reduces to the SW prepotential of the gauge theory.

Besides the prepotential, the chiral dynamics is completely specified by the correlators of the adjoint chiral field Φ. They are computed by integrals of the same type we have just described with extra ϕ_I insertions. A generating function for all chiral correlators can be written as

$$\langle {\rm tr}\, e^{z\Phi} \rangle = \sum_u e^{z a_u} - Z_{\rm inst}^{-1}(q) \sum_{k=1}^{\infty} \frac{q^k}{k!} \int \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i}\ z_k(\phi) \sum_{J=1}^{k} e^{z\phi_J} \prod_{l=1}^{2} (1 - e^{z\epsilon_l}) \tag{5}$$

$SU(N)$ plus fundamental matter

The functions appearing in (2) and (5) in the case of a $SU(N)$ gauge theory with fundamental matter are

$$\begin{aligned} D(x) &= \frac{x(x+\epsilon_1+\epsilon_2)}{(x+\epsilon_1)(x+\epsilon_2)} \\ Q_0(x) &= \frac{M(x)}{P_0(x+\epsilon_1+\epsilon_2)P_0(x)} \end{aligned} \tag{6}$$

with

$$P_0(x) = \prod_{u=1}^{N} (x - a_u) \qquad M(x) = \prod_{a=1}^{N_f} (x - m_a) \tag{7}$$

The various contributions to $D(x)$ come from D(-1)D(-1) strings while those in $Q_0(x)$ come from D(-1)D3 and D(-1)D7 open strings. Each contribution accounts for one complex moduli. In particular the denominator of $D(x)$ comes from those moduli describing the position of the instanton in the four dimensional spacetime. The numerator in $D(x)$ accounts for the ADHM constraints and the $U(k)$ gauge redundance. The denominators in $Q_0(x)$ describe the D(-1)D3 moduli. Finally the contributions to $M(x)$ come from massless fermionic moduli in the D(-1)D7 sector with N_f the number of D7 branes and m_a parametrizing the masses of the corresponding fundamental matter.

$SU(N)$ plus an Adjoint hypermultiplet

The functions appearing in (2) and (5) in the case of $SU(N)$ gauge theory with adjoint matter read

$$\begin{aligned} D(x) &= \frac{x(x+\epsilon_1+\epsilon_2)(x+m+\epsilon_1)(x+m+\epsilon_2)}{(x+\epsilon_1)(x+\epsilon_2)(x+m)(x+m+\epsilon_1+\epsilon_2)} \\ Q_0(x) &= \frac{P_0(x-m)P_0(x+m+\epsilon_1+\epsilon_2)}{P_0(x)P_0(x+\epsilon_1+\epsilon_2)} \end{aligned} \tag{8}$$

with m the mass of the adjoint hypermultiplet. Now the denominator of $D(x)$ describes the position of the instantons in an eight dimensional space and the numerators are the generalized ADHM constraints. Similarly the D(-1)D3 open strings contain, besides the standard moduli connected to the instanton radius and orientations, a set of auxiliary fields contributing to the numerator of $Q_0(x)$.

3 Saddle point analysis

In this section we use a saddle point technique to determine the instanton partition function in the limit $\epsilon_1 \to 0$ with $\epsilon_2 = \epsilon$ finite. We rederive here some results of [12] in a form more suitable for various generalizations. For simplicity we take $a_u, \epsilon_{1,2}$ to be real. Our formulae will be later extended to the complex plane by supplementing $\epsilon_{1,2}$ with a small and positive imaginary part. Exponentiating the products in (2) one can write the partition function

in the form

$$Z_{\text{inst}} = \sum_{\vec{Y}} \mathrm{e}^{\sum_{I,J}{}' \ln D(\phi_I^Y - \phi_J^Y) + \sum_I \ln(qQ_0(\phi_I^Y))} \tag{9}$$

It is convenient to introduce the density function

$$\rho(x) = \epsilon_1 \sum_I \delta(x - \phi_I) \tag{10}$$

describing the distribution of the instantons along the real line. Moreover in the limit $\epsilon_1 << x$ the function $D(x) \approx 1$ as can be seen from formulae (6) or (8) above. We write

$$e^{\ln D(x)} \approx e^{\epsilon_1 G(x)} \tag{11}$$

Plugging this into (9) and using (10) one finds

$$Z_{\text{inst}} = \int D\rho\, \mathrm{e}^{\frac{1}{\epsilon_1}\mathcal{H}_{\text{inst}}(\rho)} \tag{12}$$

with

$$\boxed{\mathcal{H}_{\text{inst}}(\rho) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} dxdy \rho(x)\rho(y) G_s(x-y) + \int_{\mathbb{R}} dx \rho(x) \ln\left[qQ_0(x)\right]} \tag{13}$$

and

$$G_s(x) = G(x) + G(-x) = \lim_{\epsilon_1 \to 0} \frac{1}{\epsilon_1} \ln D(x) D(-x) \tag{14}$$

twice the even part of $G(x)$. The main contribution to the partition function will come from instanton configurations $\rho(x)$ extremizing (13). We remark that the Z_{pert} we introduced earlier does not depend on the instanton density $\rho(x)$ and therefore it is irrelevant for the discussion of the saddle point.

3.1 The density function

The integral in (9) runs over the density functions of type (10) with ϕ^Y specified by the corresponding Young tableaux set. According to (3), in the limit $\epsilon_1 \to 0$, the instantons form a continuous distribution starting at

$$x_{ui}^0 = a_u + (i-1)\epsilon \qquad u = 1, \ldots N \quad i = 1, \ldots \infty \tag{15}$$

and ending at some x_{ui}, given by the top end of the i^{th} column in Y_u. This implies in particular that the sequence x_{ui} decreases when i grows (keeping fixed u) reaching $x_{ui} = x^0_{ui}$ at some i where the Young tableau ends. The instanton position ϕ^Y_I can then be described in terms of the continuous variable

$$\phi^Y_I = \phi_{u,i} \in [x^0_{ui}, x_{ui}] \tag{16}$$

The sums over I can then be written as

$$\sum_I = \frac{1}{\epsilon_1} \sum_{ui} \int_{x^0_{ui}}^{x_{ui}} d\phi_{ui} \tag{17}$$

and the density function (10) becomes [4]

$$\rho(x) \quad = \quad \sum_{ui} \left[\theta(x - x^0_{ui}) - \theta(x - x_{ui})\right] = \begin{cases} 1 & x \in [x^0_{ui}, x_{ui}] \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

It is important to stress that the set $\{x_{ui}\}$ completely specifies the Young tableaux set and therefore the density function $\rho(x)$. In particular the empty Young tableaux set corresponds to taking $x_{ui} = x^0_{ui}$, i.e. $\rho(x) = 0$. In Figure 1 we display (in green) the instanton distribution along the ϕ line associated to a given Young tableau profile. The columns of this tableau are to be thought of as made out of a large number of thin boxes ending at the points x_{ui} determined by the saddle point equations.

3.2 Saddle point equations

The leading contribution to the integral (9) in the limit $\epsilon_1 \to 0$ comes then from a configuration $\{x_{ui}\}$ extremizing (13)

$$\boxed{\tfrac{\delta \mathcal{H}_{\rm inst}(\rho)}{\delta x_{ui}} = \textstyle\int_{\mathbb{R}} dy \rho(y) G_s(x_{ui} - y) + \ln\left[q Q_0(x_{ui})\right] = 0} \tag{19}$$

The function $G_s(x)$ defined by (14) can be conveniently written in the form (see formulae (33) and (56) below)

$$G_s(x) = \sum_{a=1} (-)^a \frac{d}{dx} \ln(x + \alpha_a) \tag{20}$$

[4] Here we use $\int_a^b \delta(x-y)dy = -\theta(x-y)\Big|_a^b$ with $\theta(x)$ the Heaviside step function.

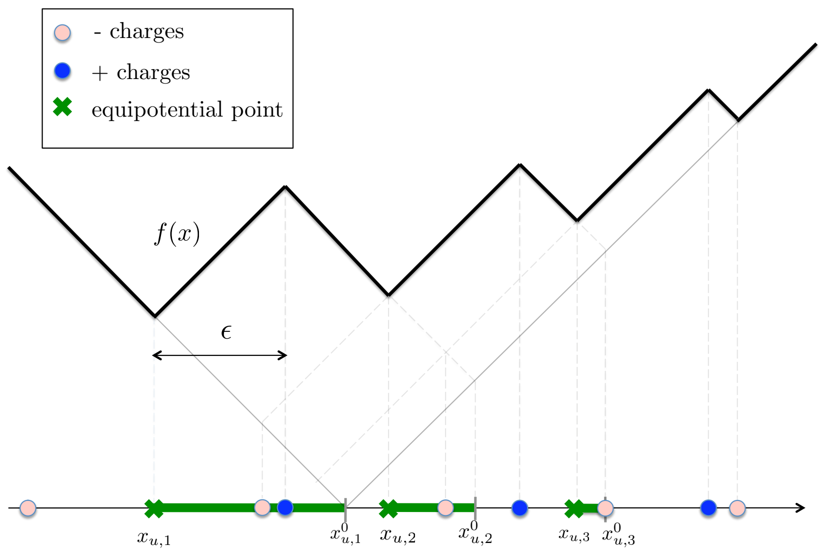

Figure 1: Instanton distribution associated to a Young tableau centered at a_u. We display in green the intervals of non trivial instanton density, i.e. $\rho(x) = 1$. By blu and red bullets we indicate plus and minus charges in the auxiliary electrostatic problem where crosses stand for points of constant potential.

for some α_a that depends on the specific matter content.

The integral in the first term in (19) becomes

$$\int_{\mathbb{R}} dy \rho(y) G_s(x-y) = \sum_{u,i} \int_{x^0_{ui}}^{x_{ui}} dy\, G_s(x-y) = \sum_{a=1} (-)^{a+1} \ln \mathcal{Y}(x+\alpha_a) \quad (21)$$

with $\mathcal{Y}(x)$ given by

$$\mathcal{Y}(x) = \prod_{v=1}^{N} \prod_{i=1}^{\infty} \left(\frac{x - x_{vi}}{x - x^0_{vi}} \right) \quad (22)$$

We notice that the function $\mathcal{Y}(x)$ has zeros at x_{ui} and poles at x^0_{ui}. The convergence of the infinite product entering in (22) follows from the fact that $x_{ui} = x^0_{ui}$ for i large enough.

It will be also convenient to define the x^0_{ui} independent ratio

$$w(x) \;=\; \frac{\mathcal{Y}(x-\epsilon)}{\mathcal{Y}(x)P_0(x)} = \lim_{L\to\infty} \frac{1}{(-L\epsilon)^N} \prod_{u=1}^{N} \prod_{i=1}^{L} \frac{x - x_{ui} - \epsilon}{x - x_{ui}} \tag{23}$$

The right hand side follows after writing the products over i in the definitions of $\mathcal{Y}, \mathcal{Y}_0$ up to a cut off $i = L$ and sending L to infinity (see Appendix A for an alternative derivation and details). Notice that all x^0_{ui} dependence cancel out in this limit. (23) shows that $w(x)$ has zeros at $x_{ui} + \epsilon$ and poles at x_{ui}.

For large x the functions $\mathcal{Y}(x)$ and $w(x)$ behave as

$$\mathcal{Y}(x) \approx 1 \qquad w(x) \approx \frac{1}{x^N} \tag{24}$$

The functions $\mathcal{Y}(x)$ or $w(x)$ encode the information about the saddle point configuration solving (19) and therefore the saddle point equation can be viewed as a functional equation for one of these functions. In particular, we will see how the functional equation for $w(x)$ provides an ϵ deformed version of the familiar SW curves encoding the corrections induced by the $\Omega_{\epsilon_1,\epsilon_2}$ background on the gauge theory.

3.3 Chiral correlators

The saddle point analysis in the last section can be also applied to the integrals (5) defining the chiral correlators. Notice first that the two integrals share the same saddle point since they only differ by ϕ_I insertions. Denoting the saddle point instanton configuration and the instanton number, giving the leading contribution in the sum over k, by $\{\phi_{I,\mathrm{ext}}\}$ and k_{ext}, one finds[5]

$$\langle \mathrm{tr}\, e^{z\Phi} \rangle \approx \sum_{u=1}^{N} e^{z a_u} - (1 - e^{z\epsilon_1})(1 - e^{z\epsilon_2}) \sum_{i=1}^{k_{\mathrm{ext}}} e^{z\phi_{i,\mathrm{ext}}} \tag{25}$$

[5] At leading order $Z_{\mathrm{inst}} \approx q^{k_{\mathrm{ext}}} Z_{k_{\mathrm{ext}}}$ and there is a cancellation between the numerator and the denominator.

Keeping only the leading contribution in the $\epsilon_1 \to 0$ limit and expanding in powers of z both sides of this equation, one finds

$$
\begin{aligned}
\langle \mathrm{tr}\Phi^J \rangle &\approx \sum_{u=1}^{N} a_u^J + \int_{\mathbb{R}} dy\, \rho(y) \frac{d}{dy} \left[y^J - (y+\epsilon)^J \right] \\
&\approx \sum_{u=1}^{N} a_u^J + \sum_{u=1}^{N} \sum_{i=1}^{\infty} \left[x_{ui}^J - (x_{ui}+\epsilon)^J + (x_{ui}^0+\epsilon)^J - x_{ui}^{0\,J} \right] \quad (26)
\end{aligned}
$$

where we use the equivalent writings (10) and (18) of $\rho(x)$ in the first and second lines respectively. (26) is just what one would have obtained from the results in [23, 24] after performing the $\epsilon_1 \to 0$ limit. The sums on the r.h.s. can be written as a contour integral as

$$
\begin{aligned}
\langle \mathrm{tr}\Phi^J \rangle &= \int_{\gamma} \frac{dy}{2\pi i}\, y^J\, \partial_y \ln \left[\frac{P_0(y) \mathcal{Y}(y)}{\mathcal{Y}(y-\epsilon)} \right] \\
&= -\int_{\gamma} \frac{dy}{2\pi i}\, y^J\, \partial_y \ln w(y) \quad (27)
\end{aligned}
$$

where γ is a counterclockwise contour surrounding the whole real line. In writing this we use the fact that $\mathcal{Y}(x)$ and $P_0(x)$ have zeros of order one at x_{ui} and a_u. Moreover, $\mathcal{Y}(x)$ has poles at x_{ui}^0 which cancel against the zeroes of $P_0(x)$ leading to the x_{ui}^0 independent function, $w(x)$ in the second line of (27). The integral on the right hand side picks up the residues at all these poles. Multiplying by x^{-J-1} this equation (with x a point outside of γ), summing over J and thinking of γ as a contour integral around the single pole at $y = x$ one can write the generating function of the chiral correlators as

$$
\boxed{\langle \mathrm{tr} \tfrac{1}{x-\Phi} \rangle = -\partial_x \ln w(x)} \quad (28)
$$

On the other hand the prepotential of the gauge theory is computed using

$$
\begin{aligned}
q\frac{d\mathcal{F}}{dq} &= -\epsilon \left(q \frac{\partial \mathcal{H}_{\mathrm{inst}}}{\partial x_{ui}} \frac{\partial x_{ui}}{\partial q} + q \frac{\partial \mathcal{H}_{\mathrm{inst}}}{\partial q} \right) = -\epsilon \int_{\mathbb{R}} \rho(y) dy \\
&= -\epsilon \sum_{ui} (x_{ui} - x_{ui}^0) = -k_{ext} \epsilon_1 \epsilon \quad (29)
\end{aligned}
$$

where the saddle point equation $\frac{\partial \mathcal{H}_{\mathrm{inst}}}{\partial x_{ui}} = 0$ has been used. We notice that the prepotential is related to the number of boxes, k_{ext}, in the Young tableau

giving the leading contribution to the partition function. The expression on the right hand side can be related to $\langle \mathrm{tr}\Phi^2 \rangle$ using the equation in the second line of (26) for $J = 2$. One finds

$$\langle \mathrm{tr}\Phi^2 \rangle = \sum_u a_u^2 + 2q \frac{d\mathcal{F}}{dq} \tag{30}$$

the well known Matone relation[26]. The classical vevs are defined by

$$\boxed{a_u = -\int_{\gamma_u} \frac{dy}{2\pi i}\, y\, \partial_y \ln w(y)} \tag{31}$$

with γ_u a contour surrounding all the x_{ui} and $x_{ui} + \epsilon$ for a fixed u.

Summarizing, the prepotential and the chiral correlators of an $\mathcal{N} = 2$ gauge theory in presence of a non trivial ϵ background can be written in terms of the integrals of the ϵ deformed SW differential

$$\lambda = -x\, d \ln w(x) \tag{32}$$

with $w(x)$ encoding the details of the saddle point solution $\{x_{ui}\}$ of (19). In the next sections we will rewrite the saddle point equations as functional equations for $w(x)$ that reduce to the SW curves in the limit of $\epsilon \to 0$.

4 $SU(N)$ plus fundamental matter

In this section we specify our general formulae to the case of $SU(N)$ with fundamental matter. The ϵ deformed SW curve in this case was derived in [12]. The solution for the U(1) case was found and written in terms of hypergeometric functions. Here we review these results and present a solution for the $SU(N)$ case. We also show that the saddle point equation can be interpreted as a " non commutative" version of the "standard" SW curve with ϵ measuring the scale of non commutativity.

4.1 Saddle point equations

From (6) and (14) one finds

$$\begin{aligned} G_s(x) &= \frac{d}{dx} \ln \left(\frac{x+\epsilon}{x-\epsilon} \right) \\ Q_0(x) &= \frac{M(x)}{P_0(x) P_0(x+\epsilon)} \end{aligned} \tag{33}$$

Notice that in this case α_a defined in (20) assumes the values

$$\alpha_a = (-\epsilon, \epsilon) \tag{34}$$

Plugging this into (21) and using (23) and (33) one finds

$$\begin{aligned}\int_{\mathbb{R}} dy\, G_s(x-y)\,\rho(y) &= \ln \frac{\mathcal{Y}(x-\epsilon)}{\mathcal{Y}(x+\epsilon)} \\ &= \ln\left(\frac{w(x)w(x+\epsilon)M(x)}{Q_0(x)}\right)\end{aligned} \tag{35}$$

The saddle point equation (19) then becomes

$$\boxed{1 - q\, M(x_{ui})\, w(x_{ui})\, w(x_{ui}+\epsilon) = 0} \tag{36}$$

This equation can be solved recursively for x_{ui} order by order in q. Indeed, at order q^L, one can write

$$x_{ui} = a_u + (i-1)\epsilon + \sum_{k=i}^{L} \lambda_{ui;k}\, q^k \tag{37}$$

and approximate $w(x)$ by $w_L(x)$ given by restricting the products over i up to $i = L$

$$w(x) \approx w_L(x) = \frac{1}{P_0(x-L\epsilon)} \prod_{u=1}^{N} \prod_{i=1}^{L} \frac{x - x_{ui} - \epsilon}{x - x_{ui}} \tag{38}$$

Plugging (37) and (38) into (36) and solving for $\lambda_{ui;k}$ one finds the x_{ui} characterizing the saddle point solution. Here we will follow an alternative route and extract the function $w(x)$ from a functional equation (see next section) that generalizes the SW curve to finite ϵ.

We conclude this section with a comment on the electrostatic interpretation of the saddle point equation. To this aim, we notice that (36) can be rewritten as

$$U(x_{ui}) = \text{const} \tag{39}$$

with

$$U(x) = \sum_{vj} \ln\left[\frac{x - x_{vj} - \epsilon}{x - x_{vj} + \epsilon}\right] + \sum_{a=1}^{N_f} \ln(x - m_a) \tag{40}$$

The function $U(x)$ can be interpreted as the two dimensional electrostatic potential generated by a set of positive charges at $x_{ui}+\epsilon$, m_a and negative charges at $x_{ui}-\epsilon$. The saddle point equation, (39), is nothing else that the condition that the potential at all points x_{ui} has the same value. We illustrate in figure 1 the electrostatic problem for $N_f=0$ and a generic Young tableau diagram with $L=3$ columns. Positive and negative charges are displayed as blue/black and pink/grey bullets. The net charge is -2, independently of the value of L.

4.2 Deformed SW differential

(36) can be rewritten as a holomorphic equation in the complex plane by introducing a function $f(x)$ defined as

$$f(x)=\frac{1-q\,M(x-\epsilon)w(x)w(x-\epsilon)}{w(x)} \tag{41}$$

It is easy to see that $f(x)$ has no poles since the zeros of the denominator at $x=x_{ui}+\epsilon$ are also zeros of the numerator according to the saddle point equation (36). In addition at large x, using the asymptotics (24), one finds $f(x)\approx x^N$ for $N_f<2N_c$ and $f(x)\approx(1-q)x^N$ for $N_f=2N_c$[6]. We can then conclude that function $f(x)$ is a polynomial of order N. We write

$$f(x)=P(x)=\prod_u(x-e_u) \tag{42}$$

in the case of $N_f<2N_c$ and $f(x)=(1-q)P(x)$ for $N_f=2N_c$. For $N_f<2N_c$ the saddle point equation becomes

$$\boxed{q\,M(x-\epsilon)\,w(x)w(x-\epsilon)+w(x)P(x)-1=0} \tag{43}$$

For $N_f=2N_c$, equation (43) still holds after replacing $P(x)\to(1-q)P(x)$. At $\epsilon=0$, equation (43) reduce to the familiar SW curve for SU(N) with matter in the fundamental representation.

The meaning of the parameters e_u can be understood considering the limit of large x where $w(x)$ behaves as

$$w(x)=\sum_{i=0}^{\infty}c_i x^{-N-i} \tag{44}$$

[6]For this choice the theory is conformal and q is dimensionless.

Plugging this into (43) and solving for the first few c_i's one finds

$$-\partial_x \ln w(x) = \sum_{J=0}^{2N-N_f-2} \sum_{u=1}^{N} \frac{e_u^J}{x^{J+1}} + o(x^{-2N+N_f}) \tag{45}$$

i.e., using (28)

$$\langle \mathrm{tr}\phi^J \rangle = \sum_{u=1}^{N} e_u^J \qquad J < 2N - N_f \tag{46}$$

This implies that the parameters e_u specifying the polynomial $P(x)$ can be interpreted as the quantum analog of the v.e.v.'s a_u[25].

A solution of the deformed SW (43) can be easily written in a continuous fraction form

$$\begin{aligned} w(x) &= \frac{1}{P(x) + q\, M(x-\epsilon)\, w(x-\epsilon)} \\ &= \frac{1}{P(x) + \frac{q\, M(x-\epsilon)}{P(x-\epsilon) + \frac{q\, M(x-2\epsilon)}{P(x-2\epsilon)+\ldots}}} \end{aligned} \tag{47}$$

Expanding in powers of q one finds $w(x) = \sum_k w_k(x) q^k$ with

$$\begin{aligned} w_0(x) &= \frac{1}{P(x)} \\ \frac{w_1(x)}{w_0(x)} &= -\frac{M(x-\epsilon)}{P(x-\epsilon)P(x)} \end{aligned} \tag{48}$$

and so on. Plugging this into (31) one finds

$$a_u = -\sum_{i=0}^{\infty} \mathrm{Res}_{y=e_u+i\epsilon}\, y\, \partial_y \ln w(y) \tag{49}$$

$$= e_u - \frac{q}{P'(e_u)} \left[\frac{M(e_u-\epsilon)}{P(e_u-\epsilon)} + \frac{M(e_u)}{P(e_u+\epsilon)} \right] + O(q^2) \tag{50}$$

Similarly, chiral correlators and the prepotential can be computed plugging $w(x)$ into (27) and (30).

4.3 Quantizing the SW curve

We conclude this section by showing that the ϵ deformed equations just obtained can be derived from a "non commutative" version of the SW curve. To this aim, we write the undeformed SW curve for the $SU(N)$ gauge theory with fundamental matter in the form

$$W(x,e^z) = q\,M(x)\,e^{-z} + P(x) - e^z = 0 \tag{51}$$

In these coordinates the SW differential reads $\lambda = xdz$. The variable z is related to the w function entering the definition of the deformed SW differential via $z = -\ln w$.

We claim that the ϵ deformed dynamics can be extracted from the eigenvalue equation

$$W(\hat{x}, e^{\hat{z}})|\Psi\rangle = 0 \tag{52}$$

that follows from promoting the SW coordinates to non commutative variables satisfying

$$[\hat{z}, \hat{x}] = \epsilon \tag{53}$$

Taking $\hat{z} = \epsilon\partial_x$, $\hat{x} = x$ and $|\Psi\rangle = \Psi(x)$, the eigenvalue problem (52) reduces to the difference equation

$$\boxed{q\,M(x)\Psi(x-\epsilon) + P(x)\Psi(x) - \Psi(x+\epsilon) = 0} \tag{54}$$

where we used $e^{\pm\hat{z}}\Psi(x) = \Psi(x\pm\epsilon)$. Dividing by $\Psi(x+\epsilon)$ and identifying

$$w(x) = \frac{\Psi(x-\epsilon)}{\Psi(x)} \tag{55}$$

one reproduces the deformed SW curve (43) after the trivial redefinition of the parameters e_u or equivalently the replacement $P(x) \to P(x+\epsilon)$.

It is worth noting the close relation of (54) to the Baxter's T-Q equation which emerges in the context of 2d integrable models. In fact for the pure $U(N)$ case with no extra matter ($M(x) \equiv 1$) this difference equation exactly coincides with the Baxter's equation for the periodic N particle Toda chain [27]. We are not aware of similar relations in the case of matter in the fundamental representation.

5 $SU(N)$ plus adjoint matter

In this section we specify to the $\mathcal{N} = 2$ gauge theory with an adjoint hypermultiplet.

5.1 The saddle point equations

From (8) and (14) one finds

$$\begin{aligned} G_s(x) &= \frac{d}{dx}\ln\left(\frac{(x+\epsilon)(x+m)(x-m-\epsilon)}{(x-\epsilon)(x-m)(x+m+\epsilon)}\right) \\ Q_0(x) &= \frac{P_0(x-m)P_0(x+m+\epsilon)}{P_0(x+\epsilon)P_0(x)} \end{aligned} \tag{56}$$

Notice that the parameters α_a in (20) are given now by

$$\alpha_a = (-\epsilon, \epsilon, -m, m, m+\epsilon, -m-\epsilon) \tag{57}$$

Plugging this into (21) one finds

$$\begin{aligned} \int_{\mathbb{R}} dy\, G_s(x-y)\,\rho(y) &= \ln\frac{\mathcal{Y}(x-\epsilon)\mathcal{Y}(x-m)\mathcal{Y}(x+m+\epsilon)}{\mathcal{Y}(x+\epsilon)\mathcal{Y}(x+m)\mathcal{Y}(x-m-\epsilon)} \\ &= \ln\left(\frac{w(x)w(x+\epsilon)}{w(x+m+\epsilon)w(x-m)\,Q_0(x)}\right) \end{aligned} \tag{58}$$

The saddle point equation (19) then becomes

$$\boxed{1 - q\frac{w(x_{ui})w(x_{ui}+\epsilon)}{w(x_{ui}+m+\epsilon)w(x_{ui}-m)} = 0} \tag{59}$$

(59) can be solved again perturbatively for x_{ui} with the help of (37) and (38).

We notice that the saddle point equations can be reinterpreted again as the equipotential condition (39) for the two dimensional electrostatic potential

$$U(x) = \sum_{vj}\ln\left[\frac{(x-x_{vj}-\epsilon)(x-x_{vj}-m)(x-x_{vj}+m+\epsilon)}{(x-x_{vj}+\epsilon)(x-x_{vj}+m)(x-x_{vj}-m-\epsilon)}\right] \tag{60}$$

Now the charges distribute near a_u and $a_u \pm m$. The net charge around a_u is -2 as in the case of a pure $\mathcal{N}=2$ gauge theory (see figure 1). Similarly one can see that the net charges around $a_u \pm m$ is $+1$[7].

[7] Alternatively one can think of $U(x)$ as the superposition $U(x) = U_m(x) + U_{-m}(x+\epsilon)$ with $U_m(x) = \sum_{vj}\ln\left[\frac{(x-x_{vj}-\epsilon)(x-x_{vj}-m)}{(x-x_{vj})(x-x_{vj}-m-\epsilon)}\right]$. $U_m(x)$ is the potential generated by a set of plus-minus charges with net charge -1 and +1 at a_u and a_u+m respectively.

5.2 Deformed SW differential

Like in the case of fundamental matter, matching the m independent zeros and poles on the right hand side of the saddle point equation (59), one can write

$$1-q\frac{w(x)w(x-\epsilon)}{w(x+m)w(x-m-\epsilon)}=w(x)P(x)f_m(x) \tag{61}$$

with $f_m(x)$ an unknown function with only m dependent zeros and poles. Indeed at large m this equation reduces to that of the pure $SU(N)$ gauge theory case after rescaling $qm^{2N}\to q$ and setting $f_m\to 1$. This implies in particular that $f_m(x)$ has only m dependent zeros and poles as claimed.

Taking the log of both sides of (61), multiplying by $\frac{1}{z-x}$ and integrating over a contour γ including all the zeros and poles of $w(x)$, but not z, we can convert this equation into an integral equation for $w(x)$

$$\boxed{\ln w(x)=-\ln P(x)+\int_\gamma \frac{dz}{2\pi i(x-z)}\,\ln\left(1-q\frac{w(z)w(z-\epsilon)}{w(z+m)w(z-m-\epsilon)}\right)} \tag{62}$$

where we use the fact that $f_m(x)$ has no zeros or poles inside γ in order to discard its contribution to the integral in the right hand side. This equation can be easily solved perturbatively in q, writing $w(x)=\sum_{k=0}^{\infty}w_k(x)q^k$ and solving recursively for $w_k(x)$. The integral reduces then to a sum over residues at $x=e_u+i\epsilon$ with $i=0,1,...$ For the first few terms one finds [8]

$$\begin{aligned}w_0(x)&=\frac{1}{P(x)}\\ \frac{w_1(x)}{w_0(x)}&=1-\frac{P(x+m)P(x-m-\epsilon)}{P(x)P(x-\epsilon)}\end{aligned} \tag{63}$$

The chiral correlators and the prepotential are given in terms of this function via (27) and (30). In particular one finds

$$\begin{aligned}a_u&=-\sum_{i=0}^{\infty}\mathrm{Res}_{z=e_u+i\epsilon}\,z\,\partial_z\ln w(z) \qquad (64)\\ &=e_u-\frac{q}{P'(e_u)}\sum_{\kappa=\pm}\frac{P(e_u-\kappa\,m)P(e_u+\kappa\,m+\kappa\,\epsilon)}{P(e_u+\kappa\,\epsilon)}+O(q^2)\end{aligned}$$

[8] In finding $w_1(x)$ we deform the contour γ to include $z=x$ and $z=\infty$ and compute the corresponding residues.

5.3 Quantizing the curve

In this section we consider the non commutative version of the SW curve for the case of the $SU(N)$ gauge theory with adjoint matter. The SW curve in this case is still given by a polynomial of order N in x but now the coefficients of the polynomial are given by modular functions involving infinite powers of $e^{\pm z}$. On the other hand, as we have seen in the last section, the deformed SW differential is found as a solution of an integral rather than a difference equation. The connection between the two descriptions is far from obvious and unlike the case of matter in the fundamental representation we are not able to build the complete dictionary. Still, we will show that the non commutative version of the SW curve again captures the physics of the ϵ deformation. In particular we will show that the prepotential of the SU(2) theory with adjoint matter is reproduced by the difference equation following from promoting the coordinates (x, z) to non commutative variables. A more complete analysis of this case deserves further investigations.

The SW curve for the $SU(N)$ gauge theory with adjoint matter can be written in the compact form [28] [9]

$$W(x, e^z) = \vartheta_1 \left(z - m\partial_x\right) P(x) = 0 \tag{65}$$

with

$$\vartheta_1(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} q^{\frac{r^2}{2}} \, e^{(z-\pi i)r} \tag{66}$$

and $P(x)$ a polynomial of order N. A non commutative version of this curve can be written by taking $z \to \epsilon\partial_x$ and introducing the wave function $\Psi(x)$ annihilated by the curve, i.e.

$$\Psi(x)\, \vartheta_1 \left(\epsilon \overleftarrow{\partial}_x - m \overrightarrow{\partial}_x\right) P(x) = 0 \tag{67}$$

with the arrows indicating whether the derivatives act on $P(x)$ or on $\Psi(x)$. Expanding on m one finds

$$\boxed{\sum_{r\in\mathbb{Z}+\frac{1}{2}} \sum_{n=0}^{N} e^{\pi i r}\, q^{\frac{r^2}{2}}\, \Psi(x - r\epsilon)\, \frac{r^n}{n!}\, (m\partial_x)^n P(x) \, = 0} \tag{68}$$

[9] Explicitly $W(x, e^z) = \sum_{n=0}^{N} \frac{1}{n!} \frac{\vartheta_1^{(n)}(z)}{\vartheta_1(z)} (m\,\partial_x)^n P(x)$, with $\vartheta_1^{(n)}(z) = \partial_z^n\, \vartheta_1(z)$.

Dividing by $\Psi(x+\frac{\epsilon}{2})$ this equation can be rewritten entirely in terms of the ratio

$$y(x) = \frac{\Psi(x-\frac{\epsilon}{2})}{\Psi(x+\frac{\epsilon}{2})} \tag{69}$$

The resulting difference equation can be easily solved perturbatively order by order in q, writing $y(x) = \sum_k y_k(x) q^k$ and solving for $y_k(x)$.

Example: $SU(2)$ gauge theory

We illustrate our analysis in the case of the $SU(2)$ gauge theory. From (68), taking $P(x) = x^2 - u$ and diving by $\Psi(x+\frac{\epsilon}{2})$ one finds for the first few orders

$$\begin{aligned} 0 &= \left[(x-\tfrac{m}{2})^2 - u\right] - y(x)\left[(x+\tfrac{m}{2})^2 - u\right] \qquad (70) \\ &\quad + q\left(y(x)y(x-\epsilon)\left[(x+\tfrac{3m}{2})^2 - u\right] - \frac{\left[(x-\frac{3m}{2})^2 - u\right]}{y(x+\epsilon)} \right) + O(q^3) \end{aligned}$$

Writing $y(x) = \sum_k y_k(x) q^k$ and solving for $y_k(x)$ gives

$$\begin{aligned} y_0(x) &= \frac{(x-\frac{m}{2})^2 - u}{(x+\frac{m}{2})^2 - u} \qquad (71) \\ \frac{y_1(x)}{y_0(x)} &= -\frac{xm(m^2-\epsilon^2)\left[8(x^2-u)(3m^2-6u-2x^2+2\epsilon^2)+3m^4-12m^2\epsilon^2\right]}{4\prod_{l=0}^{1}\left[(x+\frac{m}{2}-l\epsilon)^2-u\right]\left[(x-\frac{m}{2}+l\epsilon)^2-u\right]} \end{aligned}$$

and so on. $y(x)$ has four sets of poles. The analog of the periods of the "standard" SW curve are given by

$$\begin{aligned} a &= -\int_{\gamma_u} \frac{dx}{2\pi i}(x+\tfrac{m}{2})\partial_x \ln y(x) = -\sum_{k=0}^{\infty} \mathrm{Res}_{x=\sqrt{u}-\frac{m}{2}+k\epsilon}\left[(x+\tfrac{m}{2})\partial_x \ln y(x)\right] \\ &= \sqrt{u} - q\,\frac{m^2(m+\epsilon)^2}{\sqrt{u}(4\,u-\epsilon^2)} + O(q^2) \qquad (72) \end{aligned}$$

where γ_u is a contour surrounding one of the four sets of poles. The four choices are equivalent, for definitiveness we take the poles at $\sqrt{u} - \frac{m}{2} + k\epsilon$. (72) perfectly matches[10] the result (64) coming from the integral equation for the deformed differential after taking $e_u = \pm e$ and identifying

$$\langle \mathrm{tr}\Phi^2 \rangle = 2e^2 = 2u - 4m(m+\epsilon)\sum_{k|d} d\, q^k \tag{73}$$

[10] We checked the agreement till order q^3.

with the sum over the divisors d of k. (73) is the ϵ deformed version of the result in [29]. This provides a rather convincing evidence that the non commutative deformation of the SW curve captures the physics of the ϵ corrections of the $\mathcal{N} = 2^*$ theory. A complete proof of this equivalence would be very welcome.

Acknowledgments

We thank D. Fioravanti, A. Okounkov and O.Ragnisco for useful discussions and N. Dorey for correspondence. This work was partially supported by the ERC Advanced Grant n.226455 *"Superfields"*, by the Italian MIUR-PRIN contract 20075ATT78, by the NATO grant PST.CLG.978785, European Commission FP7 Programme Marie Curie Grant Agreement PIIF2-GA-2008-221571 and Institutional Partnership grant of the Humboldt Foundation of Germany.

A Counting functions: The toolkit

In this appendix we collect some properties of the counting functions $\mathcal{Y}(x)$ and $w(x)$ appearing in the text. These functions can be written in terms of the entire functions Y, Y_0 defined as

$$\begin{aligned} Y(z) &= e^{\frac{z}{\epsilon}\sum_u \psi\left(\frac{a_u}{\epsilon}\right)} \prod_{vj} \left(1 - \frac{z}{x_{vj}}\right) e^{\frac{z}{x^0_{vj}}} \\ Y_0(z) &= e^{\frac{z}{\epsilon}\sum_u \psi\left(\frac{a_u}{\epsilon}\right)} \prod_{vj} \left(1 - \frac{z}{x^0_{vj}}\right) e^{\frac{z}{x^0_{vj}}} \end{aligned} \tag{74}$$

with

$$x^0_{ui} = a_{ui} + (i-1)\epsilon \tag{75}$$

and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Functions $Y(z)$and $Y_0(z)$ are holomorphic with zeros in x_{ui} and x^0_{ui} respectively. They generalize the familiar infinite products entering

in the definition of the Γ function

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\, e^{-\frac{z}{n}} = \frac{e^{-\gamma z}}{\Gamma(z+1)}$$
$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n+a}\right)\, e^{-\frac{z}{n+a}} = \frac{\Gamma(1+a)}{\Gamma(1+a+z)} e^{\psi(a+1)z} \tag{76}$$

with γ the Euler-Mascheroni constant. The second equation follows from the first one after using

$$\psi(a+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+a}\right) \tag{77}$$

The second equation in (76) gives an alternative writing for the Y_0 function as

$$Y_0(z) = \prod_v \frac{\Gamma\left(\frac{a_v}{\epsilon}\right)}{\Gamma\left(\frac{a_v - z}{\epsilon}\right)} \tag{78}$$

and implies

$$\frac{Y_0(z)}{Y_0(z-\epsilon)} = \frac{(-)^N P_0(z)}{\epsilon^N} \tag{79}$$

Then one finds

$$\begin{aligned}
\mathcal{Y}(x) &= c\,\frac{Y(x)}{Y_0(x)} \qquad\qquad c = \prod_{ui}\frac{x_{ui}}{x^0_{ui}} \\
w(x) &= \frac{\mathcal{Y}(x-\epsilon)}{\mathcal{Y}(x)P_0(x)} = \frac{(-)^N}{\epsilon^N}\frac{Y(x-\epsilon)}{Y(x)}
\end{aligned} \tag{80}$$

where the last equation follows from (79). The saddle point equations for the case of fundamental and adjoint matter can be written in terms of Y, Y_0 in the form

$$0 = 1 - \frac{q\, M(x_{ui})}{\epsilon^{2N}}\,\frac{Y(x_{ui}-\epsilon)}{Y(x_{ui}+\epsilon)} \tag{81}$$

$$0 = 1 - q\,\frac{Y(x_{ui}-\epsilon)Y(x_{ui}-m)Y(x_{ui}+m+\epsilon)}{Y(x_{ui}+\epsilon)Y(x_{ui}+m)Y(x_{ui}-m-\epsilon)} \tag{82}$$

Alternatively one can define a finite size version of the functions $\mathcal{Y}(x)$, $w(x)$

$$\mathcal{Y}_L(x) = \prod_{u=1}^{N}\prod_{i=1}^{L}\frac{x_{ui}-x}{x^0_{ui}-x} \tag{83}$$
$$w_L(x) = \frac{\mathcal{Y}_L(x-\epsilon)}{\mathcal{Y}_L(x)P_0(x)} = \frac{1}{P_0(x-L\epsilon)}\prod_{u=1}^{N}\prod_{i=1}^{L}\frac{x-x_{ui}-\epsilon}{x-x_{ui}}$$

where in the last line we made use of the identity

$$\prod_{u=1}^{N}\prod_{i=1}^{L}\frac{x-x^0_{ui}}{x-x^0_{ui}-\epsilon} = \frac{P_0(x)}{P_0(x-L\epsilon)} \tag{84}$$

The last equation in (83) gives an alternative definition for the $w(x)$ counting function

$$w(x) = \lim_{L\to\infty}\frac{1}{(-L\epsilon)^N}\prod_{u=1}^{N}\prod_{i=1}^{L}\frac{x-x_{ui}-\epsilon}{x-x_{ui}} \tag{85}$$

B The effective Hamiltonian from the profile function

In this appendix we present an alternative derivation of the saddle point equations starting from the partition function written as an integral over the profile function $f(x)$ describing the Young tableaux. The function $f(x)$, as it was the case for $\rho(x)$, is completely determined in terms of the x_{ui} and the use of one or the other is just a matter of tastes.

In [6] it was showed that the ϵ deformed partition function for the pure $SU(N)$ gauge theory can be written as

$$Z(\epsilon_1,\epsilon_2;\Lambda) = \int Df\, e^{-\frac{1}{\epsilon_1\epsilon_2}\mathcal{H}_{\epsilon_1,\epsilon_2}(f)} \tag{86}$$

with

$$\mathcal{H}_{\epsilon_1,\epsilon_2}(f) = \frac{\epsilon_1\epsilon_2}{4}\int dxdy f''(x)f''(y)\gamma_{\epsilon_1,\epsilon_2}(x-y;\Lambda) \tag{87}$$

The function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$ is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda) = \frac{d}{ds}\Big|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}\frac{t^s e^{-tx}}{(e^{t\epsilon_1}-1)(e^{t\epsilon_2}-1)} \tag{88}$$

and $f(x)$ is a piecewise constant function

$$\begin{aligned} f(x) &= \sum_u |x-a_u| + \sum_{ui} \big(|x-x_{ui}-\epsilon_1| - |x-x_{ui}-\epsilon_1-\epsilon_2| \\ &\qquad -|x-x_{ui}^0| + |x-x_{ui}^0-\epsilon_2|\big) \end{aligned} \tag{89}$$

describing the profile of the Young tableaux set $\{Y_u\}$ (see figure 1). The profile function is specified by the set $\{x_{ui}\}$, with x_{ui} describing the height of the i^{th} column of the Young tableau Y_u centered at a_u.

The saddle point equation follows from extremizing $\mathcal{H}_{\epsilon_1,\epsilon_2}(f)$ with respect to f and reads

$$\int dy f''(y) \left[\gamma''_{\epsilon_1,\epsilon_2}(x-y;\Lambda) + \gamma''_{\epsilon_1,\epsilon_2}(y-x;\Lambda)\right] = 0 \tag{90}$$

where the variation is taken over piecewise functions of type (89). In the limit $\epsilon_1\epsilon_2 \to 0$, the profile function $f(x)$ becomes a smooth function and the saddle point equation (90) reduces to[11]

$$\int dy f''(y) \log\left|\frac{x-y}{\Lambda}\right| = 0 \qquad \text{for} \quad x \in \Sigma_u \tag{91}$$

where Σ_u is an interval around a_u where $f''(x)$ is non zero. Notice that the right hand side of this equation can be interpreted as the two dimensional electrostatic potential generated by a continuous charge distribution $f''(x)$ along Σ_u. The saddle point equation becomes then the condition that the potential is constant along Σ_u, i.e. Σ_u can be thought as a metallic plate[12]. The electrostatic problems are illustrated in figure 2 for the case of $SU(2)$ gauge theory with $N_f = 3$ and adjoint matter. The map $z(x)$ sends the real line to a domain in the complex plane in such a way that its imaginary is constant along the plates, while its real part is constant on the path with no electric charges. This domain is shown on the right hand side of figures (a) and (b). The explicit form of $z(x)$ is given by the Christoffel formula

$$z(x) = -\log w(x) = \int^x \prod_a (y-y_a)^{\varphi_a/\pi - 1} dy \tag{92}$$

[11] Here we use $\lim_{\epsilon_\ell \to 0} \epsilon_1\epsilon_2 \gamma_\epsilon(x;\Lambda) \approx -\frac{x^2}{2}\log\left(\frac{x}{\Lambda}\right) + \frac{3}{4}x^2$.

[12] The extreme points of Σ_u are determined by solving the electrostatic problem.

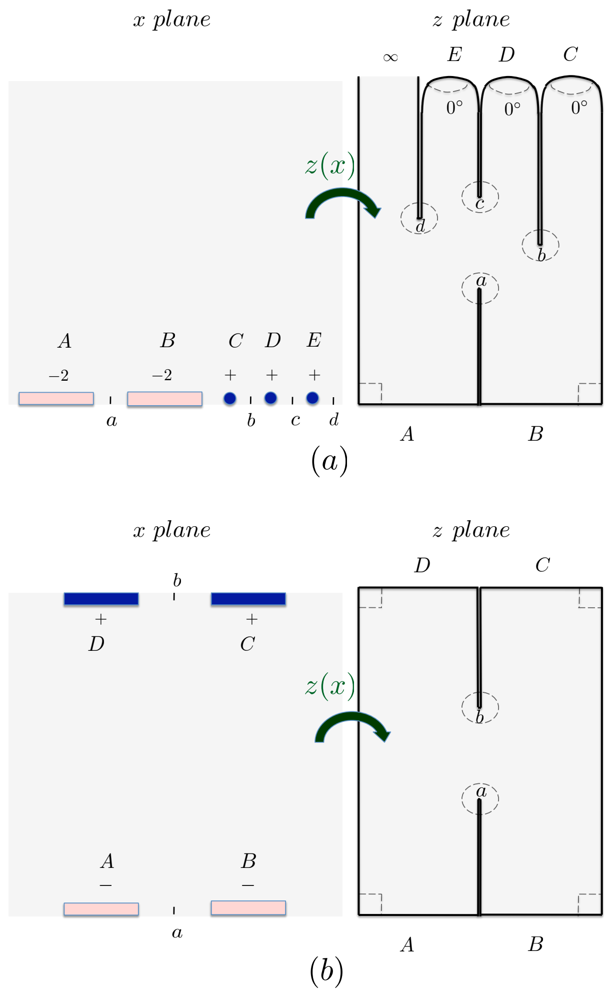

Figure 2: Electrostatic problem associated to the $\epsilon_1, \epsilon_2 \to 0$ limit of $SU(2)$ gauge theory with $N_f = 3$ fundamental matter (a) or one Adjoint hypermultiplet (b).

This map is specified by the points y_a and angles φ_a of the polygon in the right hand side of figures (a) and (b). The integrand $dz(x)$ in this formula is related to the SW differential.

In this paper we study the saddle point equations for $\epsilon_1 \to 0$ with $\epsilon_2 = \epsilon$ finite. This can be thought as a discretization of the electrostatic problem we have just exposed where the metallic plates split into an infinite number

of plus-minus dipoles and the right hand side polygons in the figure are replaced by polygons with only vertical lines (all charges are point like). The Hamiltonian become

$$\mathcal{H}_\epsilon(f) \equiv \lim_{\epsilon_1 \to 0} \mathcal{H}_{\epsilon_1,\epsilon_2}(f) \tag{93}$$

In this limit the γ function reduces to

$$\begin{aligned} \gamma_\epsilon(x;\Lambda) &= \lim_{\epsilon_1\to 0} \epsilon_1 \gamma_{\epsilon_1,\epsilon_2}(x;\Lambda) \frac{d}{ds}\Big|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty dt\, \frac{t^{s-2}e^{-tx}}{(e^{t\epsilon}-1)} \\ &= \epsilon \frac{d}{ds}\left[\frac{\Lambda^s}{(s-1)\epsilon^s}\zeta(s-1, 1+\tfrac{x}{\epsilon})\right]_{s=0} \end{aligned} \tag{94}$$

with $\zeta(s,x)$ the Hurwitz zeta function. The propagator $\gamma''(x)$ entering in the saddle point equations becomes[13]

$$\gamma_\epsilon''(x;\Lambda) \frac{d}{dx} \log\Gamma(1+\tfrac{x}{\epsilon}) + \frac{1}{\epsilon}\log\frac{\Lambda}{\epsilon} \tag{95}$$

In addition the profile function (89) reduces to

$$\begin{aligned} f''(x) &= 2\sum_u \delta(x-a_u) + 2\sum_{ui}\left[\delta(x-x_{ui}) - \delta(x-x_{ui}-\epsilon)\right. \\ &\qquad \left. -\delta(x-x_{ui}^0) + \delta(x-x_{ui}^0-\epsilon)\right] \\ &= 2\sum_{ui}\left[\delta(x-x_{ui}) - \delta(x-x_{ui}-\epsilon)\right] \end{aligned} \tag{96}$$

where in the second line we used the relation $x_{ui}^0 + \epsilon = x_{u,i+1}^0$ to cancel x_{ui}^0-dependent terms against the $\delta(x-a_u)$ contributions. Plugging (95) and (96) into (90) one finds

$$\frac{d}{dx}\log\prod_{vj}\frac{(x-x_{vj}+\epsilon)}{(x-x_{vj}-\epsilon)} = 0 \qquad \text{for} \quad x = x_{ui} \tag{97}$$

in agreement with (36) after taking $M(x) = 1$.

We remark that Z contains both perturbative and non-perturbative contributions to the instanton partition function. The perturbative part can

[13] Here we use the following Zeta function identities: $\zeta(-1,x) = \frac{1}{12} + \frac{x}{2}(1-x)$, $\partial_z\partial_x\zeta(z,x)\Big|_{z=-1} = \log\Gamma(x) + x - \frac{1}{2}(\log 2\pi + 1)$.

be easily extracted by taking $q=0$, i.e. $x_{ui}=x^0_{ui}$. In this limit one finds $f''(x)=2\sum_u \delta(x-a_u)$ leading to

$$Z_{\rm pert}=e^{-\frac{1}{\epsilon_1}\sum_{u,v}\gamma_\epsilon(a_u-a_v;\Lambda)} \tag{98}$$

In the case of matter in the fundamental and adjoint representations [6, 30, 31], $\mathcal{H}_{\epsilon_1,e_2}$ can be treated in a similar way. In these cases the additional contributions to the Hamiltonian are given by

$$\begin{aligned}
\mathcal{H}_{\rm fund}(f) &= -\tfrac{\epsilon_1\epsilon_2}{2}\sum_a\int dx f''(x)\gamma_{\epsilon_1,e_2}(x+m_a;\Lambda) \qquad (99)\\
\mathcal{H}_{\rm adj}(f) &= -\frac{\epsilon_1\epsilon_2}{4}\int dxdy f''(x)f''(y)\gamma_{\epsilon_1,\epsilon_2}(x-y+m;\Lambda)+\tfrac{\log q}{4}\int dx x^2 f''(x)
\end{aligned}$$

Following the same manipulations as before one can easily reproduces the saddle point equations (36) and (59). The matter contributions to the perturbative partition function read

$$\begin{aligned}
Z_{\rm fund,pert}(f) &= e^{\frac{1}{\epsilon_1}\sum_{a,u}\gamma_\epsilon(a_u+m_a;\Lambda)}\\
Z_{\rm adj,pert}(f) &= e^{\frac{1}{\epsilon_1}\left(\sum_{u,v}\gamma_\epsilon(a_u-a_v+m;\Lambda)-\frac{\log q}{2}\sum_u a_u^2\right)}
\end{aligned} \tag{100}$$

C Testing the deformed SW differentials

In this appendix we test the prepotential following from the deformed SW differential against a direct computation based on the partition function. We restrict ourselves to $k=1$.

C.1 SU(N) plus fundamental matter

The $k=1$ SW prepotential in the limit $\epsilon_1\to 0$ is given by

$$\begin{aligned}
\mathcal{F}_1 &= -\lim_{\epsilon_1\to 0}\epsilon_1\epsilon_2 Z_1=-\epsilon\int_{\mathbb{R}}\frac{dx}{2\pi i}\frac{M(x)}{P_0(x+\epsilon)P_0(x)}\\
&= -\epsilon\sum_u\frac{M(a_u)}{P_0(a_u+\epsilon)P_0'(a_u)}
\end{aligned} \tag{101}$$

On the other hand using (30) and the results (50) coming from the deformed SW curve one finds

$$\begin{aligned} 2\sum_{k=1}^{\infty} k\mathcal{F}_k q^k &= \sum_u (e_u^2 - a_u^2) \\ &= 2q\sum_u \frac{a_u}{P_0'(a_u)}\left[\frac{M(a_u)}{P_0(a_u+\epsilon)} + \frac{M(a_u-\epsilon)}{P_0(a_u-\epsilon)}\right] + O(q^2) \\ &= -2q\epsilon\sum_u \frac{M(a_u)}{P_0(a_u+\epsilon)P_0'(a_u)} + O(q^2) \end{aligned} \tag{102}$$

in agreement with (101) for $k=1$. In deriving the last equation we use the identities

$$\begin{aligned} \sum_u \left[\frac{M(a_u)}{P_0'(a_u)P_0(a_u+\epsilon)} + \frac{M(a_u-\epsilon)}{P_0'(a_u)P_0(a_u-\epsilon)}\right] &= 0 \\ \sum_u \left[\frac{(a_u+\epsilon)M(a_u)}{P_0'(a_u)P_0(a_u+\epsilon)} + \frac{a_u M(a_u-\epsilon)}{P_0'(a_u)P_0(a_u-\epsilon)}\right] &= 0 \end{aligned} \tag{103}$$

that follow from the vanishing of the contour integral around infinity of the function $\frac{(x+\epsilon)^a M(x)}{P_0(x+\epsilon)P_0(x)}$ with $a=0,1$.

C.2 SU(N) plus adjoint matter

The $k=1$ SW prepotential in the limit $\epsilon_1 \to 0$ is given by

$$\begin{aligned} \mathcal{F}_1 &= -\lim_{\epsilon_1\to 0}\epsilon_1\epsilon_2 Z_1 = -\epsilon\int_{\mathbb{R}} \frac{dx}{2\pi i}\frac{P_0(x+m+\epsilon)P_0(x-m)}{P_0(x+\epsilon)P_0(x)} \\ &= -\epsilon\sum_u \frac{P_0(a_u+m+\epsilon)P_0(a_u-m)}{P_0(a_u+\epsilon)P_0'(a_u)} \end{aligned} \tag{104}$$

On the other hand using (30) and the results (64) coming from the deformed SW curve one finds

$$\begin{aligned} 2\sum_{k=1}^{\infty} k\mathcal{F}_k q^k &= \sum_u (e_u^2 - a_u^2) \\ &= 2q\sum_u \frac{a_u}{P'(a_u)}\sum_{\kappa=\pm}\frac{P(a_u-\kappa\, m)P(a_u+\kappa\, m+\kappa\,\epsilon)}{P(a_u+\kappa\,\epsilon)} + O(q^2) \\ &= -2q\epsilon\sum_u \frac{P_0(a_u+m+\epsilon)P_0(a_u-m)}{P_0(a_u+\epsilon)P_0'(a_u)} + O(q^2) \end{aligned} \tag{105}$$

in agreement with (104) for $k = 1$. In deriving the last equations we follow similar manipulations of the contour integral as in the case of fundamental matter.

D A TBA like equation

In [10] an intriguing correspondence between the dynamics of $\mathcal{N} = 2$ gauge theories on the $\Omega_{\epsilon_1,\epsilon_2}$ deformed background and quantum integrable systems was proposed. According to this proposal the deformed prepotential $\mathcal{F}(\epsilon, a_u)$ of the gauge theory is identified with the Yang Yang function of the quantum integrable model with ϵ playing the role of the Planck constant. Finally the eigenvalues of the integrable Hamiltonians $\mathcal{H}_J$ are given by the chiral correlators $\langle \mathrm{tr}\Phi^J \rangle$ after quantizing a_u by requiring $\frac{\partial \mathcal{F}}{a_u} = n_u \in \mathbb{Z}$.

In this section we show that the deformed SW equations we found in the text can be rewritten in a Thermodynamic Bethe Ansatz (TBA) like form of the type presented in [10]. For concreteness we focus on the $\mathcal{N} = 2^*$ gauge theory associated, according to [10], to the quantum elliptic Calogero-Moser system. The analysis here can be easily adapted to the case of fundamental matter. We first rewrite (59) in the Bethe Ansatz like form

$$\boxed{e^{-\varphi(x_{ui})} = 1} \tag{106}$$

with

$$-\varphi(x) \quad = \quad \ln\left(q\frac{w(x)w(x+\epsilon)}{w(x+m+\epsilon)w(x-m)}\right) \tag{107}$$

Using (62) we can rewrite each $\ln w(x + A)$ in the right hand side of (107) in an integral form. Collecting the various pieces one finds the TBA like equation

$$\boxed{-\varphi(x) = \ln qQ(x) + \int_\gamma dy\, \tilde{G}(x-y) \ln(1 - e^{-\varphi(y)})} \tag{108}$$

with

$$\begin{aligned} \tilde{G}(x) &= \frac{d}{dx}\ln\left[\frac{x(x-\epsilon)}{(x+m)(x-m-\epsilon)}\right] \\ Q(x) &= \frac{P(x+m+\epsilon)P(x-m)}{P(x)P(x+\epsilon)} \end{aligned} \tag{109}$$

(108) is close to that found in [10]. The two are related by the replacement $\tilde{G}(x) \to G_s(x)$. We don't fully understand the origin of this discrepancy. We should remark however that, despite the similarities, (106) and (108) are very different from the standard Bethe Ansatz and TBA equations . The main difference being that, unlike in the standard setting of a Bethe Ansatz, in the present case $e^{-\varphi}$ is not an unimodular phase and therefore $1 - e^{-\varphi}$ has not only zeros at $x_{ui} + \epsilon$ but also poles at x_{ui}. This explains the appearance of $\tilde{G}(x)$ rather than $G_s(x)$ in our TBA like equation (108), in contrast with the expectations coming from integrable systems where the reflection matrix is a unimodular phase. It would be nice to understand, may be along the lines of [32, 33], how TBA techniques extend to this case.

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